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Combinatorial formulae for one-dimensional generalised random walks

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Abstract. The calculation of ensemble averages in one-dimensional walks with weights given by $\exp[-g\sum(n_i)^\alpha]$, where n_i is the number of times a site is visited, is simplified by reducing the problem to one of calculating binomial factors for partitions and their permutations. The combinatorial factors are generalised to the case of Cayley trees and all lattices without closed loops. In one dimension the combinatorial formulae are shown to reduce the known cases when $\alpha=0$ and $\alpha=1$. We also use these formulae to extend the exact enumeration results, for the one-dimensional generalised random walk, from 21 to 29 terms; and the series are analysed using Padé approximant techniques.

1. Introduction

Random walk models are relevant to many physical situations. Two fundamental examples are the use of the ordinary random walk (ORW) to model Brownian motion, and the use of the self-avoiding walk (SAW) to model the behaviour of polymer chains in dilute solution (for reviews see Barber and Ninham (1970), Whittington (1982), Weiss and Rubin (1982) and for recent applications of random walks in random environments see Weiss (1983)).

Over the years, and especially recently, several generalisations of the ORW and SAW have appeared in the literature. The motivation for the introduction of the generalised walks has been varied, and in the main the generalised walks have been treated separately. In a previous paper Duxbury *et al* (1984) made a comparative study of several generalised random walks with a view to determining the quantities which affect the asymptotic properties of these models. It was suggested that, in addition to dimensionality, the important quantities in the weight given to an interacting random walk are:

(a) the range of the memory;

(b) whether the memory is cumulative or not;

(c) whether the normalisation condition in the walk is local (static) or global (kinetic).

In a second paper, Duxbury and de Queiroz (1985) made these ideas quantitative by introducing a generalised static random walk with infinite-range memory. The properties of this model were studied using effective-medium arguments and exact enumeration methods. In dimensions greater than one it was found that, for repulsive correlations, Flory-like arguments predict exponents which vary continuously with the parameter α (see equation (1) in § 2). In one dimension they found that, for attractive correlations, the model exhibits anomalous trapping for $g>0$ and $1\geqslant\alpha>0$.

In this paper we further study the generalised random walk introduced in Duxbury and de Queiroz (1985). It is shown that the problem of calculating ensemble averages for the one-dimensional random walk can be reduced to calculating partitions into the number of steps on each bond, and associating with them multiplicities which are calculated from binomial factors on all permutations of each partition. This analogy is explained in § 2. In this section it is also shown how our formulae reduce to known cases when $\alpha=0$ (the interacting walk of Stanley *et al* (1983), asymptotically solved by Redner and Kang (1983)), and $\alpha=1$ (the ORW case). In § 2 we further show that the combinatorial analysis used on the one-dimensional chain can be generalised to the case of Cayley trees and all lattices without closed loops.

In § 3, the combinatorial formulae derived in § 2 are used to perform exact enumerations on the one-dimensional generalised random walk. To carry out these calculations we need a computer algorithm for generating all ways of putting m indistinguishable objects in $S-1$ boxes. The approach discussed in § 2 allows us to extend the exact enumeration results from the 21 terms, found in the previous paper (Duxbury and de Queiroz 1985), to 29 terms. We also refine the analysis by using Padé techniques as well as the Neville tables used previously. These analyses support the effective-medium prediction that for $g>0$ and $1\geqslant\alpha>0$ the end-to-end distance exponent, and the span exponent, vary continuously with α.

The final section, § 4, gives our conclusions.

2. Derivation of the combinatorial formulae

The generalised random walk introduced in Duxbury and de Queiroz (1985) and further studied in this work, is one in which each of the 2^N walks of N steps on a one-dimensional chain are weighted with a probability given by,

$$P_{\text{walk}}=\exp\left(-g\sum_{i=1}^{S+1}(n_i)^{\alpha}\right) \tag{1}$$

where n_i is the number of times the ith site is visited during the walk, S (the span) is the total number of different bonds visited, and g and α are variable parameters. In order to study the properties of an esemble of walks with weights (1), we study the generating function (or partition function),

$$Z=\sum_{\{\text{walks}\}}P_{\text{walk}} \tag{2}$$

the average span (maximum distance from left to right) of a walk,

$$\langle S_N\rangle=Z^{-1}\sum_{\{\text{walks}\}}S_{\text{walk}}P_{\text{walk}} \tag{3}$$

(we note that the number of distinct sites visited is $S_{\text{walk}}+1$, and in fact use this quantity in the series analysis) and the average squared end-to-end distance of a walk,

$$\langle R_N^2\rangle=Z^{-1}\sum_{\{\text{walks}\}}R_{\text{walk}}^2P_{\text{walk}}. \tag{4}$$

In (2)–(4) the sum over {walks} goes over all 2^N N-step walks on a one-dimensional chain.

In the previous work (Duxbury and de Queiroz 1985) the calculations were performed by enumerating all 2^N possible walks and storing them according to which partition, on the total number of steps, they corresponded. Here we show that the problem of performing the ensemble averages in (2)-(4) can, in essence, be reduced to finding all ways of partitioning M ($=(N-2S+R)/2$, where R is the end-to-end distance of the walk, S is the span and N is the total number of steps) objects into S boxes, and then calculating a product of binomial factors for each possible arrangement of each partition on M. The total number of distinct configurations which need be considered in calculating (2)-(4) is reduced from 2^N to a number which is bounded above by $N \times 2^{2N/3}$.

To see how the reduction in the number of configurations considered arises, consider a one-dimensional random walk of N steps, with range R and span S. A walk with specified span S and range R has to perform a minimum number of steps to satisfy these conditions; this minimum walk forms the backbone of the walk. An example for a walk with range two and span five is shown in figure 1. After laying down the backbone (figure 1(a)), $2M = N-2S+R$ steps remain to be distributed. These remaining $2M$ steps may be laid down in pairs, but otherwise at random, amongst the S bonds available on the backbone. However for each partition and permutation on M ($=(N-2S+R)/2$) there may be many associated random walks. Indeed it is this degeneracy which effects the reduction in the number of configurations that need be enumerated in a series analysis. Consider the backbone in figure 1(a) and the $\{0, 1, 0, 1, 0\}$ partition of figure 1(b) placed on it (where each number in the partition denotes the number of *pairs* of steps placed on that bond). The random walks represented by this backbone and this partition then have 12 steps in all. The four different random walks associated with this backbone and partition are shown in figure 1(c). In this example the four distinct walks arise because there are two ways of

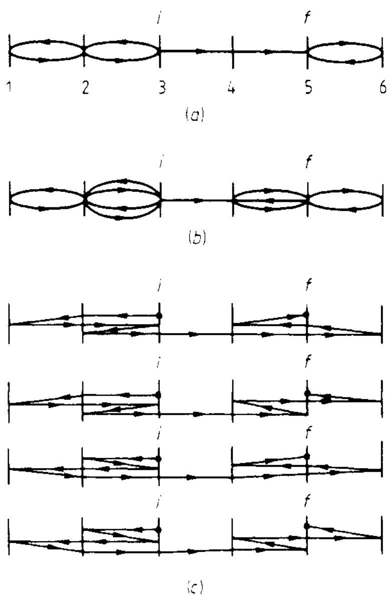

Figure 1. 12 step, one-dimensional random walks with span five and range two. (a) The backbone, (b) the dressed backbone, (c) the four different random walks corresponding to the partition $\{0, 1, 0, 1, 0\}$.

ordering the vertex at site 2 and two ways of ordering the vertex at site 5. In fact total number of different random walks associated with a backbone and partition is a product of factors at each internal vertex of the graph. It then remains to find the factor appropriate to a general vertex.

For a general vertex there are fixed constraints in that the initial entrance to, and final exit from, the vertex are specified from the neighbouring vertices (if the vertex being considered is the starting (ending) point of the walk only the final exit (initial entrance) is constrained). In addition each general exit has associated with it an entrance which ensures that the span and range constraints are satisfied. The remaining free paths exiting to the left of the vertex we call $n_{\rm l}$ and those to the right $n_{\rm r}$. The total number of different ways of ordering the entering and leaving the vertex is then

$$c_j=\binom{n_{\rm l}+n_{\rm r}}{n_{\rm l}}=\binom{n_{\rm l}+n_{\rm r}}{n_{\rm r}}. \tag{5}$$

As an example consider a vertex at site j to the left of the initial site i, and where the final site f is to the right of i, with two pairs of bonds to the left of the vertex and three pairs of bonds to the right of the vertex. The constraint is that the initial approach to, and the final exit from, the vertex must be to the right; so $n_{\rm l}=2$ and $n_l=2$ and the combinatorial factor for the vertex is $c=\binom{4}{2}=6$. These six arrangements are illustrated in figure 2.

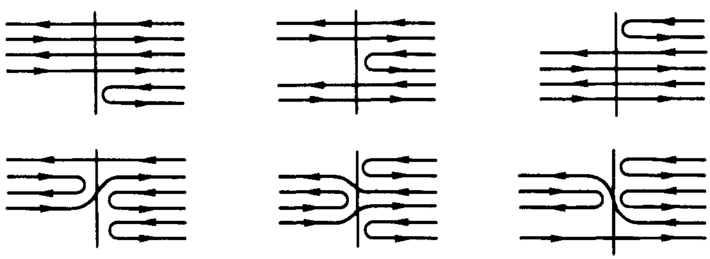

Figure 2. The six different one-dimensional random walk configurations possible at a vertex, which lies to the left of the starting site i, with two pairs of steps to the left and three pairs of steps to the right.

Thus a one-dimensional graph with N steps, range R, span S and specified partition on M has associated with it

$$C(N,R,S,\{\ \})=\prod_{k=1}^{S-1}\binom{n_{\rm l}^k+n_{\rm r}^k}{n_{\rm r}^k} \tag{6}$$

different random walks, where $n_{\rm l}^k$ and $n_{\rm r}^k$ are respectively the number of free paths to the left and right of vertex k. To be more specific, consider the graph in figure 3, which depicts walks starting at 3, ending at site 6 and with span 3. The number of walks corresponding to this diagram is calculated using equation (6) in terms of the $\{m_k\}$ as follows,

$$\begin{aligned} n_{\rm l}^k&=\begin{cases} m_k & k=1,2,\dots,f\\ m_{k-1} & k=f+1,\dots,S-1\end{cases}\\ n_{\rm r}^k&=\begin{cases} m_{k+1}-1 & k=1,\dots,i-1\\ m_{k+1} & k=i,i+1,\dots,S-1.\end{cases}\end{aligned} \tag{7}$$

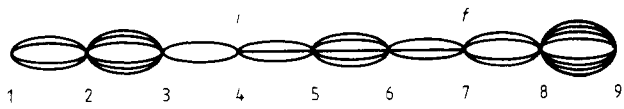

Figure 3. A graph for one-dimensional random walks with span eight, range three and partition {1, 2, 0, 1, 2, 1, 1, 3}. This graph represents 1620 different random walks.

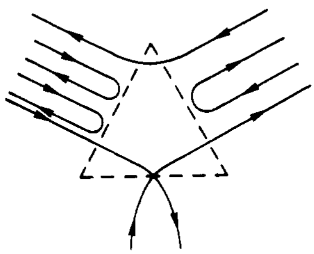

Figure 4. A random-walk vertex on a $z=3$ Cayley tree.

The resulting number of walks associated with this diagram is then

$$C=\binom{4}{2}\binom{3}{3}\binom{2}{1}\binom{3}{2}\binom{3}{1}\binom{3}{2}\binom{5}{4}=1620. \tag{8}$$

The number of partitions that must be summed for a given M and S is clearly the number of ways of putting $M=(N-2S+R)/2$ indistinguishable pairs of steps onto the S bonds of the chain. This number is easily seen to be the binomial coefficient,

$$\binom{M+S-1}{S-1}. \tag{9}$$

Therefore, summing over the span S, range R and the starting site i, we find the total number $T(N)$ of terms, which are to be evaluated in the series analysis, to be

$$T(N_{\text{even}})=\sum_{S/2=0}^{N/2}\sum_{R/2=0}^{S/2}(S-R+1)\binom{N/2+R/2-1}{S-1} + \sum_{(S-1)/2=0}^{N/2-1}\sum_{R/2=0}^{(S-1)/2}(S-R+1)\binom{N/2+R/2-1}{S-1} \tag{10}$$

where, for simplicity, we have chosen N (and therefore R) to be even, where the first sum in equation (10) is for even values of S and the second sum is over odd values of S, and where $(S-R+1)$ is the maximum number of different starting sites i, for given values S and R. The first sums over R can be easily performed exactly, using the identity,

$$\sum_{i=0}^{m}\binom{n+i}{S}=\binom{n+m+1}{S+1}-\binom{n}{S+1} \tag{11}$$

with the result that the two terms in equation (10) can be combined to give

$$T(N_{\text{even}}) = \sum_{S/2=i}^{N/2} \left[2\binom{1+(N+S)/2}{S+1} + \binom{(N+S)/2}{S} \right] - \sum_{S=1}^{N} \frac{N+(S+1)^2}{S+1} \binom{N/2-1}{S}. \quad (12)$$

The second sum diverges as $(N/8)2^{N/2}$ for large N. An upper bound can be put on the first sum by noting that $\binom{(N+S)/2}{S}$ has its maximum value at $N=3S$ and by Stirling's approximation, for large N, it takes on the value $2^{2N/3}$. Therefore, the exact number of terms to be summed numerically is reduced from 2^N to a number of the order of $2^{2N/3}$.

The combinatorial formulae for the one-dimensional walks can be easily generalised to the case of Cayley trees and lattices without loops. For example, consider a Cayley tree with $z=3$. A vertex for such a lattice is shown in figure 4. In the case depicted the walk first enters the vertex from one of the three sides and leaves from another leg, and there are n_0, n_1, n_2 free exits to be ordered in all possible ways on the three sides. The appropriate combinatorial factor for each vertex then becomes the trinomial coefficient,

$$c_j = \frac{(n_0+n_1+n_2)!}{n_0!n_1!n_2!}. \quad (13)$$

For a general vertex at site j with coordination number z (and provided the lattice has no closed loops), the appropriate factor is the multinomial coefficient,

$$c_j = \frac{(n_0+n_1+\ldots+n_{z-1})!}{n_0!n_1!n_2!\ldots n_{z-1}!}. \quad (14)$$

For the whole graph the combinatorial factor is again a product over the factors at each internal vertex of the graph.

We now indicate how the approach described above coincides, when $\alpha=0$ and $\alpha=1$, with exact results. The case $\alpha=0$ reduces to calculating the span distribution function, which can be performed exactly by a variety of methods (see for example Weiss and Rubin 1982). The case $\alpha=1$ makes all of the walk weights equal and thus is equivalent to the ORW on a one-dimensional chain (Duxbury and de Queiroz 1985). For simplicity we consider the special case of closed walks or walks on polygons ($R=0$, or $i=f$), which start and finish at site 0.

In the case of polygons, for a walk of span S, there are $2S$ steps needed for the backbone, so that the addition of M pairs of steps to the S bonds makes $N=2M+2S$ total steps. The partition function becomes, using equations (1) and (6)

$$Z_N = \sum_S^N \int_{-\infty}^{\infty} \frac{dk\,\exp(-ikM)}{2\pi} \sum_{m_1}\sum_{m_2}\ldots\sum_{m_s}\prod_{i=0}^{S}\binom{p_i}{m_i}\exp[ikm_i - g(n_i)^\alpha] \quad (15)$$

where the k integral ensures $\sum m_j = M$. Here $p_i = m_i + m_{i+1} + 1$ for all i, with $m_0 = 0$, $m_{s+1} = -1$ and $n_i = p_i + 1$.

For the special cases $\alpha=0$ and $\alpha=1$, the weight factor reduces as follows

$$\prod_{i=0}^{S} \exp[-g(n_i)^\alpha] = \begin{cases} \exp[-g(S+1)] & \text{for } \alpha=0 \\ \exp(-gN) & \text{for } \alpha=1. \end{cases} \quad (16)$$

The weight factor W can now be taken outside the $\{m_i\}$ sums and integral in (15); which may now be rewritten as

$$Z_N = \sum_S I_S W \quad (17)$$

where the sum is over all spans $S=1$ to $N/2$ and I_S is given by,

$$I_S=\int_{-\infty}^{\infty} \mathrm{d}k \exp(-\mathrm{i}kM) \sum_{m_1} \exp(\mathrm{i}km_1) \times \left[\!\left[\sum_{m_2} \exp(\mathrm{i}km_2)\binom{p_2}{m_2}\left\{\ldots\left[\sum_{m_s} \exp(\mathrm{i}km_s)\binom{p_s}{m_s}\right]\ldots\right\}\right]\!\right]. \tag{18}$$

Each of the sums in brackets can be performed using the identity

$$\sum_{m_s} A^{m_s}\binom{B+m_s+1}{B}=\frac{1}{A}\left(\frac{1}{1-A}\right)^{B+1}. \tag{19}$$

Carrying out these sums, we arrive at the following expression for I_S,

$$I_S=\int_{-\infty}^{\infty} \mathrm{d}k \frac{\exp[-\mathrm{i}k(M+S)]}{1-\cfrac{A}{1-\cfrac{A}{1-\cfrac{A}{\ddots\, \overline{1-A}}}}} \tag{20}$$

where there are S factors of $A=\exp(\mathrm{i}k)$ in the continued fraction denominator. This equation may be written as

$$I=\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k \exp[-\mathrm{i}k(M+S)]G_{00}^{s}(k) \tag{21}$$

where $G_{00}^{s}(k)$ is just the $(0,0)$ element of the Green function defined by its inverse

$$[\mathbf{G}^{s}(k)]^{-1}=\mathbf{I}_s-\exp(ik/2)\mathbf{T}_s \tag{22}$$

where $\mathbf{I}_s$ is the $(S+1)\times(S+1)$ identity matrix and $\mathbf{T}_s$ is the $(S+1)\times(S+1)$ transfer matrix for walks of maximum span S, defined by

$$\mathbf{T}_s=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & . & . & 0 \\ 1 & 0 & 1 & 0 & 0 & . & . & 0 \\ 0 & 1 & 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0 & 1 \\ 0 & . & . & . & . & 0 & 1 & 0 \end{bmatrix}. \tag{23}$$

The occurrence of the $(0,0)$ element of the Green function in the expression (21) is due to the fact that the analysis was restricted to polygons. If we had considered walks starting at i and finishing at f, the appropriate expression would have been (21) with the $(0,0)$ element of the Green function replaced by the (i,f) element. It is now clear that the above method reduces to the transfer-matrix approach of the span problem treated by many other authors (see, e.g., Weiss and Rubin 1982), and could be used to reconstruct the existing results by standard methods.

Finally, to clarify the relationship of equation (21) consider the determinant

$$|[G^s(k)]^{-1}| = \begin{vmatrix} 1 & a & 0 & 0 & 0 & . & . & 0 \\ a & 1 & a & 0 & 0 & . & . & 0 \\ 0 & a & 1 & a & 0 & . & . & 0 \\ 0 & 0 & a & 1 & a & . & . & 0 \\ . & . & . & . & . & . & 1 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 1 \end{vmatrix}. \tag{24}$$

Expanding the determinant, it is clear that, with $a = \exp(ik/2)$

$$|\mathbf{G}^{-1}| = |\mathbf{0}| = |\mathbf{1}| - a^2|\mathbf{2}| \tag{25}$$

where $|\mathbf{i}|$ is the determinant remaining after the first i rows and i columns have been deleted. In this notation, we see that the (0, 0) element of **G** is given by,

$$G^s_{00} = \frac{|\mathbf{1}|}{|\mathbf{0}|} = \frac{1}{1 - a^2|\mathbf{2}|/|\mathbf{1}|}. \tag{26}$$

However, surely $|\mathbf{2}|/|\mathbf{1}|$ can be calculated by expanding $|\mathbf{1}|$ just as in equation (25) so that after S times, we find the continued fraction given in equation (20).

3. Series generation and analysis

The series are generated for each value of g and α in equation (1) by running through all possible spans (S) and ranges (R) and for each S and R and the different starting points (the $T(N)$ terms of equation (10)), using the combinatorial analysis of the previous section. For fixed S and R there are $\binom{M+S+1}{S-1}$, where $M = (N - 2S + R)/2$, different permutations of the M pairs which may be arranged arbitrarily. This section of the procedure may be programmed efficiently by mapping each integer between 1 and $\binom{M+S-1}{S-1}$ onto a unique arrangement of a partition on M. This programming is done by letting the first bond take on all numbers of pairs of steps m_1 for $0 \leq m_1 \leq M$, then for each m_1, letting m_2 take on all values $0 \leq m_1 \leq M - m_2$, and so forth, with $0 \leq m_{i+1} \leq M - \Sigma_{j=0}^{i} m_j$. It is straightforward to show that, in general, the program steps through $\binom{M+i}{i}$ values of m_i up to the bond $i = S - 1$, which steps through the full range $\binom{M+S-1}{S-1}$ of formula (9). This results from the identity

$$\sum_{\tau=0}^{M} \binom{\tau + i - 1}{i - 1} = \binom{M + i}{i} \tag{27}$$

which is a special case of equation (11). Finally, the last bond m_s is fixed by the sum $\Sigma_{i=1}^{S} m_i = M$. Our subroutine which performs the unique transformation between an integer and a partition on M is given in the appendix.

Using the above technique the series for the one-dimensional generalised walks with weights given by equation (1), have been calculated for walks of up to 29 steps. Sample series for $\langle S_N + 1 \rangle$ (these series were used in preference to the $\langle S_N \rangle$ series as the Padé analysis shows less scatter) and $\langle R_N^2 \rangle$ at $\alpha = 0, 0.2, 0.4, 0.6, 0.8$ are given in table 1. We now use these series to further test the effective medium prediction that for $g > 0$ and $0 \leq \alpha < 1$ the model defined by equation (1) exhibits anomalous trapping with exponents which vary continuously according to $s = \nu = (1 - \alpha)/(3 - \alpha)$.

In a previous work (Duxbury and de Queiroz 1985), the ratio method was used in conjunction with Neville tables to obtain estimates of the end-to-end distance and span exponents. The results were subject to considerable scatter and it is useful to use an alternative method especially since there are now 29 terms available for analysis. Probably the most widely used extrapolation method in critical phenomena is the method of Padé approximants (see e.g. Gaunt and Guttmann 1976). It is possible to use this technique on the exact enumeration data available here by noting that if we have the sequence,

$$a_1, a_2, a_3, \ldots, a_n \tag{28}$$

which is expected to asymptotically behave as

$$a_n \simeq n^x \tag{29}$$

then its generating function has the following behaviour

$$\lim_{z\to 1} \sum_{\tau=1}^{n} a_\tau z^\tau \simeq (1-z)^{x+1}. \tag{30}$$

Table 1. Exact enumeration data for one-dimensional walks of up to 29 steps at $g = 1.0$ and $\alpha = 0$, 0.2, 0.4, 0.6, 0.8. (a) $\langle R_N^2\rangle$ series, (b) $\langle S_N + 1\rangle$ series.

(a)

N	$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
1	1.000 000 000	1.000 000 000	1.000 000 000	1.000 000 000	1.000 000 000
2	1.075 765 685	1.196 639 516	1.344 605 929	1.524 964 665	1.742 537 635
3	1.578 635 905	1.715 973 066	1.903 982 552	2.162 758 615	2.518 218 705
4	1.727 203 932	1.896 568 134	2.149 512 075	2.531 822 343	3.114 872 039
5	2.083 882 563	2.260 705 265	2.540 064 241	2.992 872 028	3.744 468 474
6	2.260 781 703	2.434 454 666	2.736 393 098	3.268 439 515	4.231 146 890
7	2.532 900 693	2.709 965 535	3.029 164 712	3.617 660 171	4.749 971 096
8	2.702 247 649	2.865 681 734	3.188 316 988	3.827 362 293	5.148 534 344
9	2.933 940 272	3.092 669 440	3.422 186 558	4.101 775 252	5.579 977 542
10	3.085 724 368	3.228 271 931	3.552 326 912	4.263 418 318	5.906 845 482
11	3.298 075 688	3.427 597 227	3.748 714 309	4.485 490 576	6.267 740 074
12	3.433 781 905	3.546 007 544	3.856 817 794	4.611 734 376	6.535 904 572
13	3.634 041 467	3.727 321 447	4.027 527 843	4.795 733 674	6.839 015 681
14	3.757 733 828	3.832 430 602	4.119 126 474	4.895 702 512	7.058 907 778
15	3.947 570 423	4.000 031 819	4.270 901 840	5.051 118 090	7.314 270 722
16	4.062 925 767	4.095 241 352	4.350 163 093	5.131 424 328	7.494 340 600
17	4.242 602 580	4.251 246 819	4.487 135 216	5.264 845 472	7.710 030 438
18	4.352 303 826	4.339 178 086	4.557 136 386	5.330 320 150	7.857 175 036
19	4.522 161 493	4.484 938 398	4.682 067 305	5.446 491 769	8.039 785 005
20	4.627 971 915	4.567 499 588	4.745 067 153	5.500 697 413	8.159 667 868
21	4.788 706 281	4.704 128 236	4.859 959 836	5.603 135 570	8.314 630 983
22	4.891 688 320	4.782 657 505	4.917 612 938	5.648 722 234	8.411 924 586
23	5.044 238 609	4.911 183 379	5.024 011 433	5.740 090 114	8.543 744 441
24	5.144 970 948	4.986 589 461	5.077 527 524	5.779 043 828	8.622 316 983
25	5.290 353 944	5.107 976 153	5.176 669 952	5.861 392 778	8.734 741 584
26	5.389 110 020	5.180 857 955	5.226 931 555	5.895 211 190	8.797 803 481
27	5.528 300 606	5.295 989 713	5.319 834 018	5.970 142 283	8.893 960 620
28	5.625 181 839	5.366 731 977	5.367 483 843	5.999 960 688	8.944 181 134
29	5.759 049 567	5.476 400 284	5.454 991 325	6.068 737 113	9.026 686 182

Table 1. (continued)

(*b*)

N	$\alpha=0$	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$
1	2.000 000 000	2.000 000 000	2.000 000 000	2.000 000 000	2.000 000 000
2	2.268 941 421	2.299 159 879	2.336 151 482	2.381 241 166	2.435 634 409
3	2.537 882 843	2.598 319 758	2.672 302 964	2.762 482 332	2.871 268 817
4	2.774 440 958	2.834 727 335	2.916 277 384	3.026 774 209	3.176 116 268
5	3.015 978 739	3.077 656 376	3.167 142 122	3.296 700 039	3.483 821 792
6	3.201 628 276	3.259 538 646	3.350 770 347	3.493 596 163	3.717 164 952
7	3.404 368 778	3.457 509 690	3.548 838 359	3.701 915 377	3.956 856 000
8	3.552 216 073	3.602 488 830	3.694 563 928	3.857 728 252	4.145 674 087
9	3.724 806 699	3.768 762 833	3.858 001 716	4.026 929 647	4.341 864 086
10	3.850 712 233	3.891 220 279	3.979 890 021	4.156 079 931	4.500 187 744
11	4.001 936 174	4.035 298 537	4.119 422 138	4.298 347 721	4.665 731 700
12	4.115 133 450	4.143 650 574	4.225 502 561	4.408 982 187	4.801 656 809
13	4.250 327 091	4.271 286 520	4.347 519 900	4.531 554 582	4.944 276 103
14	4.355 006 696	4.369 878 090	4.442 296 878	4.628 534 664	5.062 930 997
15	4.477 364 189	4.484 606 847	4.550 811 730	4.736 054 014	5.187 702 239
16	4.575 333 576	4.575 688 189	4.636 923 824	4.822 481 567	5.292 588 612
17	4.687 115 007	4.679 926 263	4.734 658 470	4.918 127 927	5.403 067 587
18	4.779 292 953	4.764 795 569	4.813 781 098	4.996 115 699	5.496 714 018
19	4.882 324 610	4.860 374 651	4.902 719 140	5.082 182 324	5.595 506 231
20	4.969 343 795	4.939 907 593	4.976 013 861	5.153 250 380	5.679 814 297
21	5.065 145 527	5.028 289 932	5.057 671 339	5.231 453 119	5.768 893 037
22	5.147 540 234	5.103 156 352	5.126 002 965	5.296 741 738	5.845 335 986
23	5.237 346 950	5.185 520 984	5.201 568 164	5.368 397 562	5.926 233 722
24	5.315 594 717	5.256 268 662	5.265 612 148	5.428 788 670	5.995 978 889
25	5.400 377 051	5.333 558 649	5.336 027 045	5.494 926 788	6.069 909 624
26	5.474 903 618	5.400 647 836	5.396 326 777	5.551 118 032	6.133 898 343
27	5.555 406 415	5.473 609 656	5.462 342 027	5.612 553 879	6.201 839 590
28	5.626 587 405	5.537 431 248	5.519 343 982	5.665 106 791	6.260 841 791
29	5.703 377 188	5.606 652 461	5.581 559 261	5.722 493 914	6.323 591 516

Both the $\langle S_N+1\rangle$ and $\langle R_N^2\rangle$ series are expected to have this behaviour, with x being the exponents s and 2ν respectively. Evaluations at $z=1$ of Padé approximants to the derivative of the log (DLOG Padé's) of the generating function provide estimates of $x+1$ and hence afford the required extrapolations to s and ν. This analysis is illustrated in tables 2 and 3 where the $z=1$ evaluations of DLOG Padé's for generalised random walk series at $\alpha=0$ and $\alpha=0.4$, and $g=1.0$ in both cases, are presented. In table 4 we give our estimates of s and ν deduced from tables 2 and 3 along with estimates of s and ν found from a similar analysis of generalised series at different values of α (all analyses were performed for series at $g=1.0$). The effective-medium prediction is included for comparison. These results are similar to those found by Duxbury and de Queiroz (1985) using Neville table extrapolants on 21-term series. Neville tables applied to the longer series available here do not show any qualitative change in behaviour. For α greater than 0.5 however, the tables do overestimate the curvature quite badly.

From table 4 it is seen that the estimates of the span exponent (s) are consistently higher than the effective medium prediction, while the estimates of the range exponent (ν) are consistently lower. Indeed the difference between ν and s is larger in this calculation than was found by Duxbury and de Queiroz (1985). This result is surprising

Table 2. $z = 1$ evaluations of Padé approximants to log derivative series at $g = 1.00$ and $\alpha = 0$. Tables gives results of $n + l/n$ Padé approximants, where $n + l$ is the order of the numerator polynomial and n is the order of the denominator polynomial. (a) $\langle R_N^2 \rangle$ series, (b) $\langle S_N + 1 \rangle$ series.

(a)

n	$l=-1$	$l=0$	$l=1$
6	1.572	1.573	1.573
7	1.573	1.573	1.576
8	1.579	1.574	1.575
9	1.575	1.573	1.561
10	1.572	1.575	1.570
11	1.570	1.571	1.570
12	1.570	1.571	1.574
13	1.575	1.557	1.594
14	1.596	1.557	

(b)

n	$l=-1$	$l=0$	$l=1$
6	1.374	1.378	1.379
7	1.379	1.378	1.255
8	1.378	1.381	1.384
9	1.408	1.437	1.377
10	1.406	1.373	1.368
11	1.371	1.369	1.372
12	1.370	1.370	1.369
13	1.373	1.368	1.370
14	1.362	1.368	

Table 3. $z = 1$ evaluations of Padé approximants to log derivative series at $g = 1.00$ and $\alpha = 0.4$. Tables give results of $(n + l)/n$ Padé approximants, where $n + l$ is the order of the numerator polynomial and n is the order of the denominator polynomial. (a) $\langle R_N^2 \rangle$ series, (b) $\langle S_N + 1 \rangle$ series.

(a)

n	$l=-1$	$l=0$	$l=1$
6	1.333	1.329	1.334
7	1.335	1.251	1.319
8	1.320	1.317	1.317
9	1.317	1.316	1.316
10	1.316	1.317	1.031
11	1.256	1.327	1.334
12	1.334	1.339	1.364
13	1.430	1.358	1.363
14	1.380	1.358	

(b)

n	$l=-1$	$l=0$	$l=1$
6	1.313	1.313	1.312
7	1.316	1.315	1.305
8	1.316	1.230	1.288
9	1.294	1.283	1.283
10	1.283	1.283	1.284
11	1.285	1.284	1.284
12	1.284	1.284	1.276
13	1.282	1.280	1.278
14	1.286	1.280	

Table 4. Estimates of the asymptotic exponents ν and s for the one-dimensional walk, with $g = 1.00$, found from the Padé analysis, compared with the effective, medium predictions.

α	ν	s	$(1-\alpha)/(3-\alpha)$
0	0.29	0.37	0.333
0.2	0.24	0.33	0.286
0.4	0.18	0.28	0.231
0.6	0.15†	0.22	0.167
0.8	0.05	0.17	0.091

† This evaluation table was badly behaved.

since more terms ($N = 29$ compared with $N = 21$) were evaluated and a Padé analysis method was used. Thus, although we still believe that our results are consistent with the effective medium predictions $\nu = s = (1-\alpha)/(3-\alpha)$, it would be useful to test the result further.

4. Conclusions

It has been shown that the problem of calculating ensemble averages in a generalised random walk, on lattices without loops and with weights given by equation (1), can be simplified by mapping the problem onto one of calculating combinatorial factors on each partition and permutation of $M = (N-2S+R)/2$ into S boxes. These formulae are shown to reduce, in appropriate limits ($\alpha=1$ and $\alpha=0$), to the ORW and the weighted-span walk. These reductions illustrate interesting connections between the transfer matrix method, the Green function approach, and a new continued fraction representation of the span distribution function.

Using the combinatorial formula derived in § 2, we have been able to extend the series for the one-dimensional generalised random walk from 21 to 29 terms. Using these series, and Padé approximant extrapolation methods, we test the effective medium prediction that for $g>0$ and $0 \leqslant \alpha < 1$ the model exhibits anomalous trapping with continuously varying exponents. The series results confirm the phenomenon of anomalous trapping, and the exponents do vary continuously. We also suggest that the series are consistent with the effective-medium prediction that $s=\nu=(1-\alpha)/(3-\alpha)$, although the results appear to suggest that $s>\nu$. This behaviour is probably due to finite lattice and crossover effects.

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Appendix

```
c      A subroutine to convert a decimal number, ICON, into a
c      set {m_i}, where {m_i} are a partition on ID. By running
c      through all integers (ICON) in between 1 and (M-1 choose IS-1),
c      PERGEN generates all partitions and permutations on ID.
c      On input ICON contains the integer, IS + 1 is the span and
c      ID = M (see text). On putput the vector NP contains the
c      partition corresponding to ICON. The array NBINC(I, J)
c      contains the binomial coefficient (I-1 choose J-1).
       SUBROUTINE PERGEN(ICON, ID, IS, NP, ISUM)
       IMPLICIT DOUBLE PRECISION(A-H, O-Y)
       COMMON/DATA/NBINC(40, 40)
       DIMENSION NP(40)
       IF(ID.EQ.0) GO TO 6
       IF(IS.EQ.1) GO TO 5
       ISUM = 0
       DO 3 J = 1, IS - 1
       ITOT1 = 0
       ITOT = 0
```

```
      DO 1 ITO=1, ID+1-ISUM
      ITOT=NBINC(ID-ISUM+IS-ITO-J+1, IS-J)+ITOT
      IF(ICON.LE.ITOT) GO TO 2
      ITOT1=ITOT
1     CONTINUE
2     NP(J)=ITO-1
      ICON=ICON-ITOT1
      ISUM=ISUM+NP(J)
      IF(ISUM.EQ.ID) GO TO 4
3     CONTINUE
4     NP(IS)=ID-ISUM
      GO TO 6
5     NP(1)=ID
6     CONTINUE
      RETURN
      END
```

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